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**Lecture 11: Constrained Optimization**

1. Mathematical indifference curves
	1. Indifference curves all have the same utility, originating from the combination of two goods. In other words, U = U(X,Y).
	2. Recall that indifference curves have requirements such as convexity (for a pair of goods which are not perfect substitutes). For any random possible utility function, how can you tell if it’s “well behaved?” For some requirements, such as continuity, it is clear. But checking for convexity and downward sloping requires a bit of math.
		1. Consider this example: U = X0.5 + Y. Is this a well-behaved utility function?
		2. First, let’s isolate Y so we can perform some operations on it. Since U is a fixed value, this is straight forward: U – X0.5 = Y.
		3. To determine if it is downward sloping, we take the first derivative: -0.5X-0.5. Since it is negative, it is downward sloping.
		4. To determine convexity, we take the derivative again, or the second derivative. That’s (-0.5)(-0.5)X-1.5, or 0.25X-1.5. Since the result is positive, the curve is convex.
	3. Here’s a slightly more difficult one: U = (XY)0.5
		1. U2 = XY

U2/X = Y

* + 1. 1st derivative: -U2/X2 √
		2. 2nd derivative: 2U2/X3 √
1. Lagrangian
	1. A *Lagrangian* is a function to maximize or minimized. We will be maximizing since we are maximizing utility.
	2. To construct it, we set up the Lagrangian, L, as so:

$$L=U\left(X,Y\right)-λ(P\_{X}X+ P\_{Y}Y-I)$$

with λ as the “Lagrangian multiplier.”

* + 1. Note that our budget constraint is set equal to zero.
	1. To determine the maximization result, we take the derivative with respect to X, Y, and λ. Then set equal to zero. Then we solve.

$$\frac{∂L}{∂X}=MU\_{X}\left(X,Y\right)-λP\_{X}=0$$

$$\frac{∂L}{∂Y}=MU\_{Y}\left(X,Y\right)-λP\_{Y}=0$$

$$\frac{∂L}{∂λ}=-P\_{X}X-P\_{Y}Y+I=0$$

* 1. We can then rewrite the results as:

$$MU\_{X}\left(X,Y\right)=λP\_{X}$$

$$MU\_{Y}\left(X,Y\right)=λP\_{Y}$$

$$P\_{X}X+P\_{Y}Y=I$$

* + 1. This last equation is not very interesting; the other two are though, because we can isolate lambda and…

$$λ=\frac{MU\_{X}\left(X,Y\right)}{P\_{X}}=\frac{MU\_{Y}\left(X,Y\right)}{P\_{Y}}$$

$$\frac{MU\_{X}\left(X,Y\right)}{MU\_{Y}\left(X,Y\right)}=\frac{P\_{X}}{P\_{Y}}$$

1. Example
	1. Suppose we set U = (XY)0.5 with the price of X being 10 and the price of Y being 5. Our budget is 30. How much do we buy of each?
		1. Note we do *not* isolate Y; that was just to determine if the curve is well behaved.

$$\frac{∂L}{∂X}=0.5Y\left(XY\right)^{-0.5}-λ10=0$$

$$\frac{∂L}{∂Y}=0.5X\left(XY\right)^{-0.5}-λ5=0$$

$$\frac{∂L}{∂λ}=-10X-5Y+30=0$$

* 1. Let’s start by isolating λ and then setting the first two equal to each other.

$$\frac{Y}{20}\left(XY\right)^{-0.5}=\frac{X}{10}\left(XY\right)^{-0.5}$$

* 1. We can then isolate either X or Y in the third equation. Let’s isolate Y and substitute.

$$\frac{6-2X}{20}\left(X(6-2X)\right)^{-0.5}=\frac{X}{10}\left(X(6-2X)\right)^{-0.5}$$

* 1. Isolating X, X = 1.5. Since X = 1.5, then Y must equal 3 (recalling our budget constraint) and λ equals about 0.071 (using either the first or second partial derivative we took).
		1. Note that this result makes sense. The utility function weighs X and Y equally but since X costs twice as much as Y, we buy half as much.